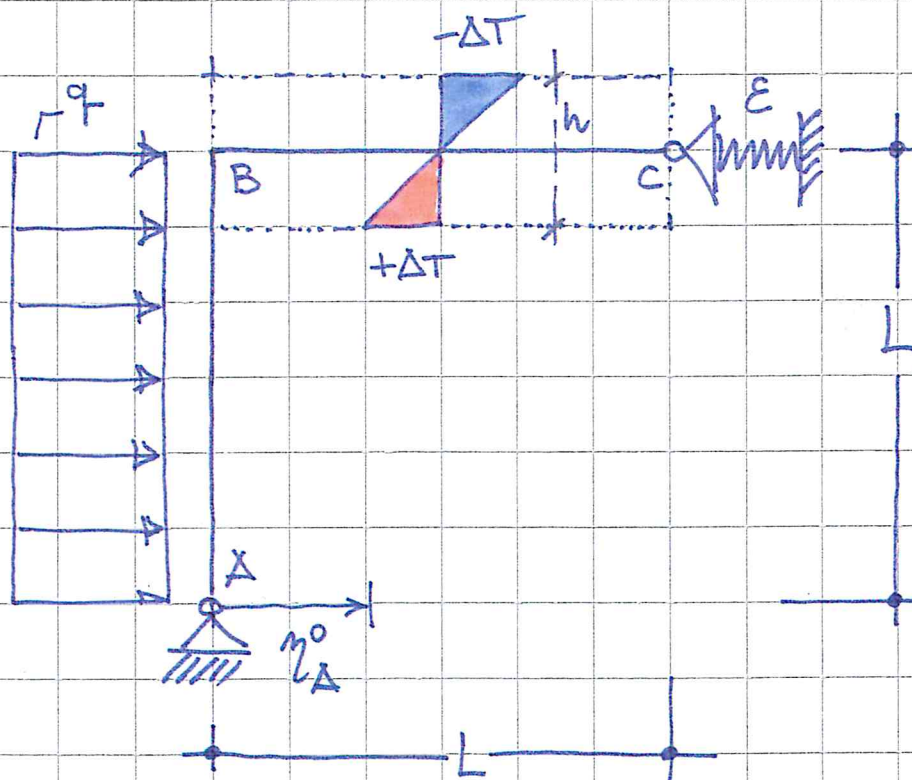


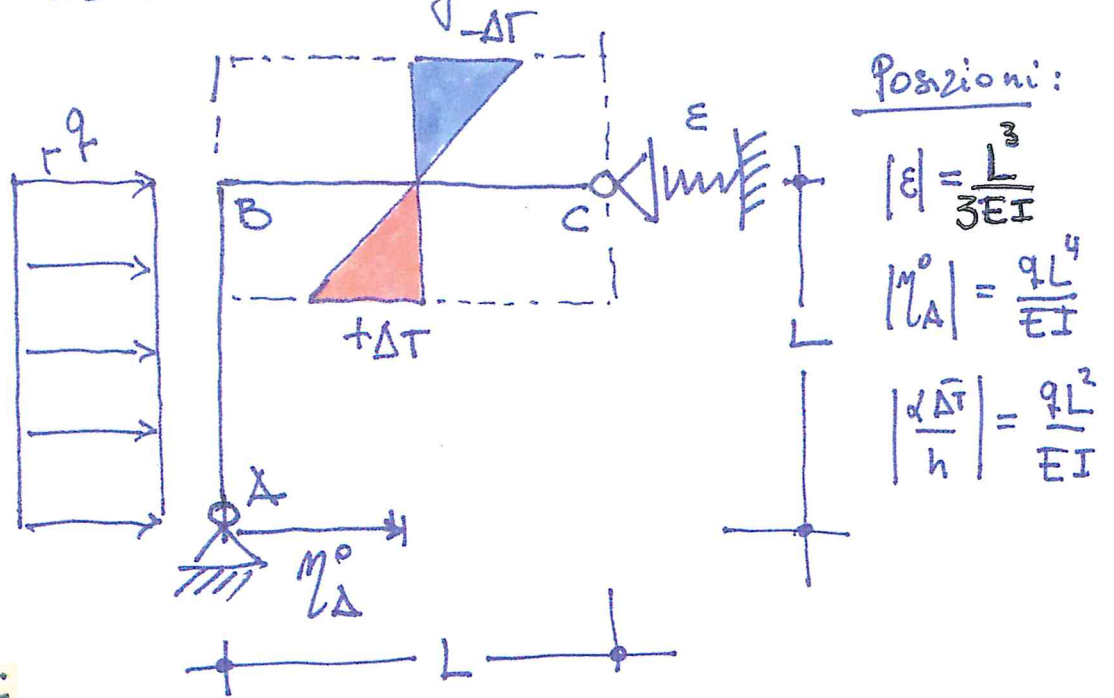
RISOLVERE LA STRUTTURA UNA VOLTA IPERSTATICA SEGUENTE -  
TRACCIARE IL DIAGRAMMA DEL MOMENTO TENENDO CONTO  
DELLE POSIZIONI RIPORTATE IN FIGURA -


$$|\varepsilon| = \frac{L^3}{3EI}$$

$$|\varphi_A^0| = \frac{qL^4}{EI}$$

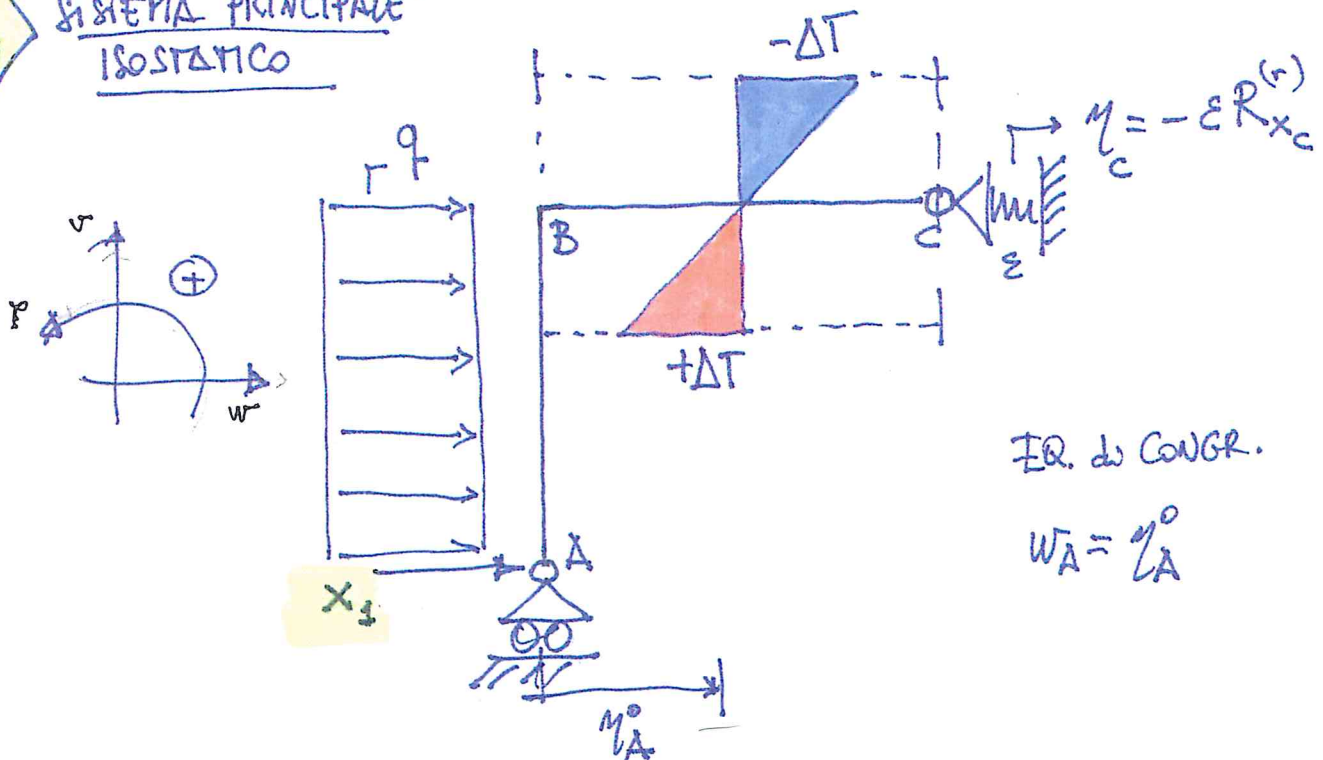
$$\left| \frac{\Delta T}{h} \right| = \frac{q L^2}{EI}$$

Risolvere la struttura una volta iperstatica riportata nella figura seguente tracciando il diagramma dei momenti.



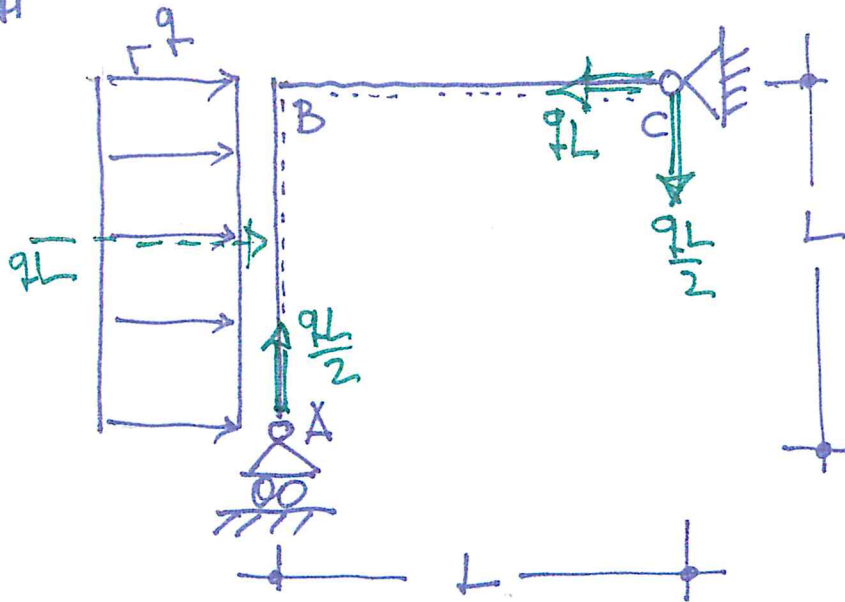
### SOLUZIONE n. 1

➡ SISTEMA PRINCIPALE  
ISOSTATICO



EQ. di CONGR.

SCHEMA [0]  
Solo CARICHI  
ESTERNI



RV  
con metodo  
grafico!

Calcolo  $M^{(0)}(z)$ :

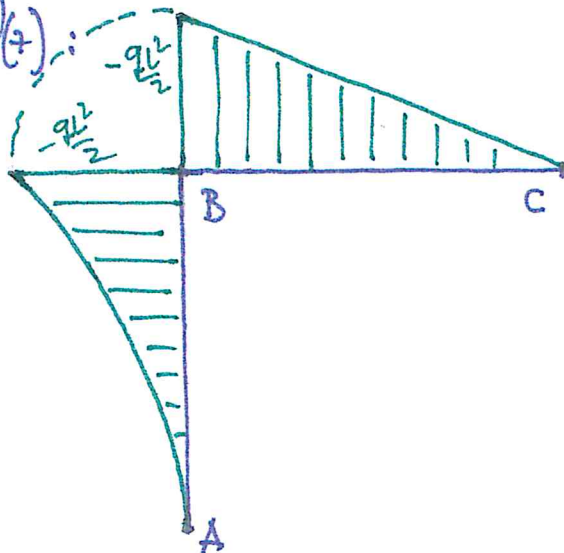
TRATTO AB  $0 \leq z \leq L$

$$M^{(0)}(z) = -\frac{qz^2}{2} \quad \left\{ \begin{array}{l} M_A = \phi \\ M_B = -\frac{qL^2}{2} \end{array} \right.$$

TRATTO BC  $0 \leq z \leq L$

$$M^{(0)}(z) = -\frac{qL}{2} [L-z] \quad \left\{ \begin{array}{l} M_B = -\frac{qL^2}{2} \\ M_C = \phi \end{array} \right.$$

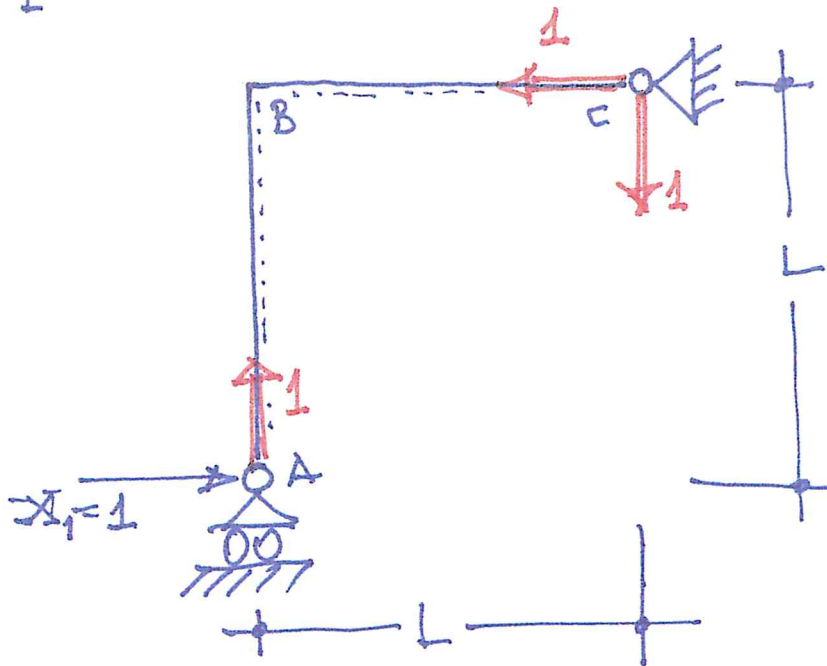
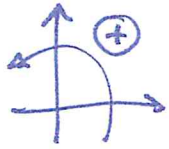
Diagramma  $M^{(0)}(z)$ :



$M^{(0)}(z)$

SCHEMA [1]

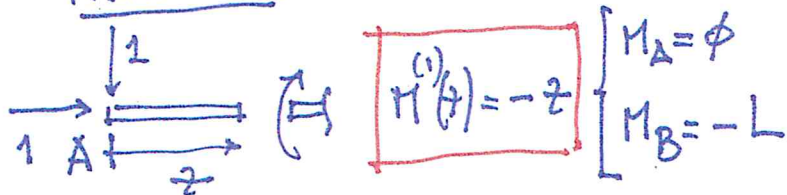
Solo  $X_1 = 1$



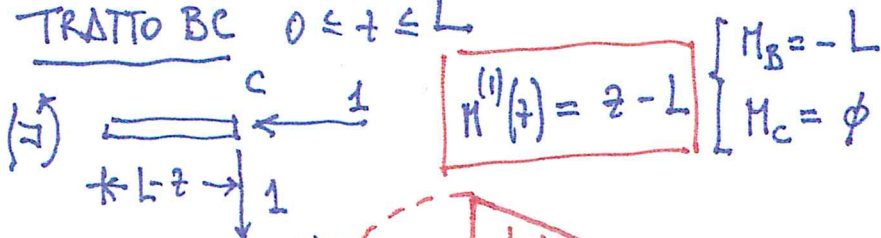
Riv  
con metodo  
grafico!

Calcoliamo  $M^{(1)}(z)$ :

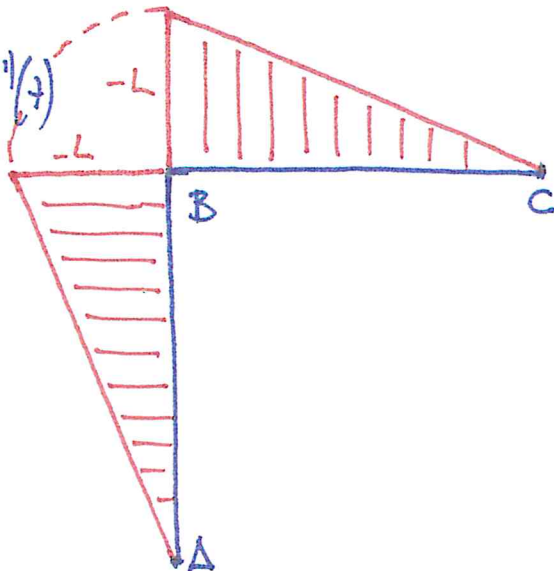
TRATTO AB  $0 \leq z \leq L$



TRATTO BC  $0 \leq z \leq L$



Disegniamo  $M^{(1)}(z)$





L'unica equazione di Müller-Breslau, corrispondente all'unica incognita ipostatica  $X_1$ , si scrive nella forma  $L_{ve} = L_{vi}$  assumendo come sistema lavorato o fittizio lo schema [1] e come sistema reale la struttura ipostatica data. Si ha:

$$\begin{aligned}
 L_{ve} &= \underbrace{1}_{X_1^{(t)}=1} \cdot \eta_i^{(r)} + \sum_j R_j^{(t)} \eta_j^{(r)} = \\
 &= 1 \cdot \eta_A^0 + \underbrace{R_{X_c}^{(1)}}_{-1} \underbrace{\eta_c^{(r)}}_{-\varepsilon R_{X_c}^{(r)}} = \\
 &\quad \underbrace{R_{X_c}^{(0)} + R_{X_c}^{(1)} X_1}_{[-qL - X_1]} \\
 &= \eta_A^0 - \varepsilon [qL + X_1]
 \end{aligned}$$

$$L_{vi} = \int_{str} M^{(t)} \frac{M^{(r)}}{EI} dstr + \int_{str} \underbrace{M^{(t)}}_{<0} \underbrace{\frac{\alpha \Delta T}{h}}_{>0} dstr =$$

$$\downarrow \text{con } M^{(r)} = M^{(0)} + M^{(1)} X_1$$

$$= \int_{str} M^{(1)} \frac{M^{(0)}}{EI} dstr + X_1 \int_{str} \frac{[M^{(1)}]^2}{EI} dstr + \int_{str} M^{(1)} \frac{\alpha \Delta T}{h} dstr =$$

$$\begin{aligned}
&= \frac{1}{EI} \left\{ \int_{\overline{AB}} [-z] \left[ -\frac{qz^2}{2} \right] dz + \int_{\overline{BC}} [z-L] \left[ -\frac{qL}{2} [L-z] \right] dz \right\} + \\
&\quad + \frac{X_1}{EI} \left\{ \int_{\overline{AB}} z^2 dz + \int_{\overline{BC}} (z-L)^2 dz \right\} + \int_{\overline{BC}} [z-L] \alpha \frac{\overline{\Delta T}}{h} dz = \\
&= \frac{1}{EI} \left\{ \int_0^L \frac{qz^3}{2} dz + \int_0^L \frac{qL}{2} \overbrace{(z-L)^2}^{z^2 + L^2 - 2Lz} dz \right\} + \\
&\quad + \frac{X_1}{EI} \left\{ \int_0^L z^2 dz + \int_0^L (z^2 + L^2 - 2Lz) dz \right\} + \alpha \frac{\overline{\Delta T}}{h} \int_0^L (z-L) dz = \\
&= \frac{1}{EI} \left\{ \frac{q}{2} \left[ \frac{z^4}{4} \right]_0^L + \frac{qL}{2} \left\{ \left[ \frac{z^3}{3} \right]_0^L + L^2 [z]_0^L - 2L \left[ \frac{z^2}{2} \right]_0^L \right\} \right\} + \\
&\quad + \frac{X_1}{EI} \left\{ \left[ \frac{z^3}{3} \right]_0^L + \left[ \frac{z^3}{3} \right]_0^L + L^2 [z]_0^L - 2L \left[ \frac{z^2}{2} \right]_0^L \right\} + \alpha \frac{\overline{\Delta T}}{h} \left\{ \left[ \frac{z^2}{2} \right]_0^L - L [z]_0^L \right\} = \\
&= \frac{1}{EI} \left[ \frac{qL^4}{8} + \frac{qL^4}{6} + \cancel{\frac{qL^4}{2}} - \cancel{\frac{qL^4}{2}} \right] + \\
&\quad + \frac{X_1}{EI} \left[ \frac{L^3}{3} + \frac{L^3}{3} + \cancel{L^3} - \cancel{L^3} \right] + \alpha \frac{\overline{\Delta T}}{h} \left[ \frac{L^2}{2} - L^2 \right] = \\
&= \frac{7}{24} \frac{qL^4}{EI} + \frac{2}{3} \frac{X_1 L^3}{EI} - \alpha \frac{\overline{\Delta T}}{h} \frac{L^2}{2}
\end{aligned}$$

In definitiva  $L_{re} = L_{vi}$  fornisce:

$$M_A^0 - \varepsilon qL - \varepsilon X_1 = \frac{7}{24} \frac{qL^4}{EI} + \frac{2}{3} \frac{L^3}{EI} X_1 - \alpha \frac{\overline{\Delta T}}{h} \frac{L^2}{2}$$

e quindi:

$$X_1 \left[ \frac{2L^3}{3EI} + \varepsilon \right] = M_A^0 - \varepsilon qL - \frac{7}{24} \frac{qL^4}{EI} + \alpha \frac{\overline{\Delta T}}{h} \frac{L^2}{2}$$

➤ Tenendo conto delle posizioni iniziali, che permettono di esprimere tutto in funzione di  $q$ ,  $L$  ed  $EI$  e cioè:

$$|\varepsilon| = \frac{L^3}{3EI} ; \quad |\gamma_A^0| = \frac{qL^4}{EI} ; \quad \left| \frac{\Delta \bar{T}}{h} \right| = \frac{qL^2}{EI}$$

La  $X_1$  può scriversi nella forma:

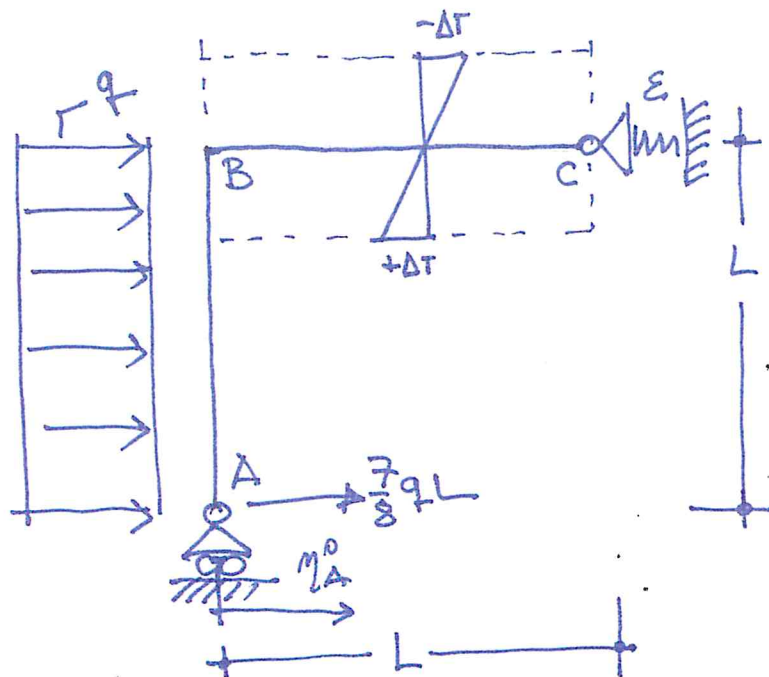
$$X_1 \left[ \underbrace{\frac{2L^3}{3EI} + \frac{L^3}{3EI}}_{\frac{L^3}{EI}} \right] = \underbrace{\frac{qL^4}{EI} - \frac{qL^4}{3EI} - \frac{7}{24} \frac{qL^4}{EI} + \frac{qL^4}{2EI}}_{\frac{qL^4}{EI} \left[ 1 - \frac{1}{3} - \frac{7}{24} + \frac{1}{2} \right]}$$

$$\frac{24 - 8 - 7 + 12}{24} = \frac{21}{24} = \frac{7}{8}$$

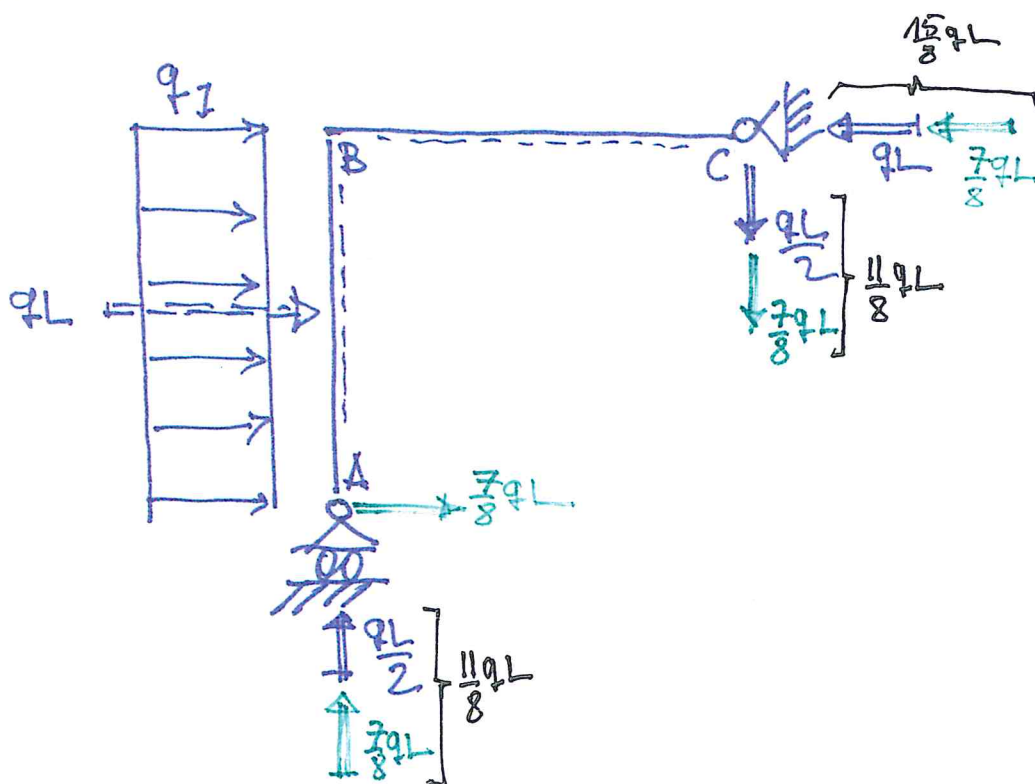
e in definitiva:

$$X_1 = \frac{7}{8} qL \quad \text{POSITIVA} \quad \Rightarrow \quad \text{VERO IPOTIZZATO CORRETTO!}$$

Il diagramma dei momenti sulla struttura iperstatica assegnata potrà determinarsi sul sistema principale ipostatico soggetto ai carichi assegnati e alla  $X_1 = \frac{7}{8} qL$  prima determinata. Si ha:



Si determinano le RV per via grafica applicando il principio di sovrapposizione degli effetti e, a seguire, la legge del momento  $M^{(r)}$ .

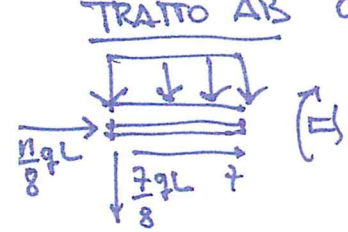


Legge di  $M(z)$  e diagrammi:

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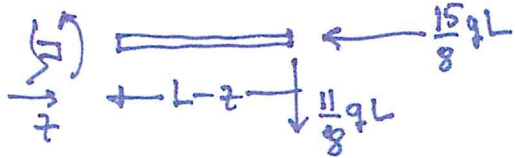


TRATTO AB  $0 \leq z \leq L$



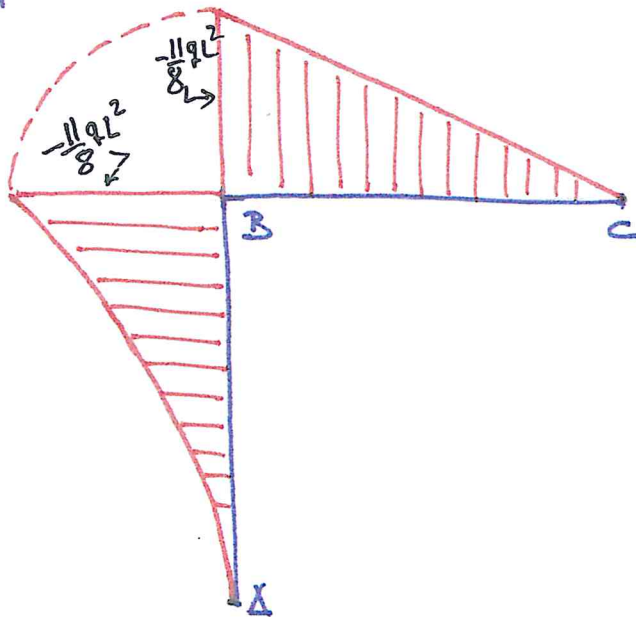
$$M(z) = -\frac{7qL}{8}z - \frac{qz^2}{2} \quad \left\{ \begin{array}{l} M_A = 0 \\ M_B = -\frac{7qL^2}{8} - \frac{qL^2}{2} = -\frac{11qL^2}{8} \end{array} \right.$$

TRATTO BC  $0 \leq z \leq L$



$$M(z) = -\frac{11qL}{8}(L-z) \quad \left\{ \begin{array}{l} M_B = -\frac{11qL^2}{8} \\ M_C = 0 \end{array} \right.$$

Si ha in definitiva:



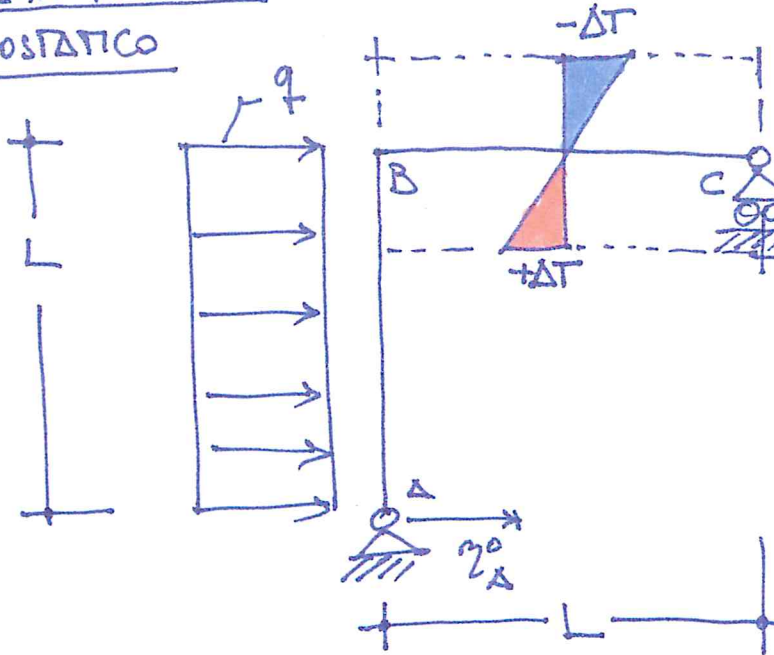
# SOLUZIONE 2

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IV

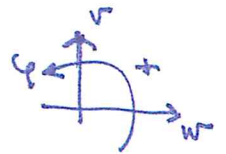


## SISTEMA PRINCIPALE ISOSTATICO

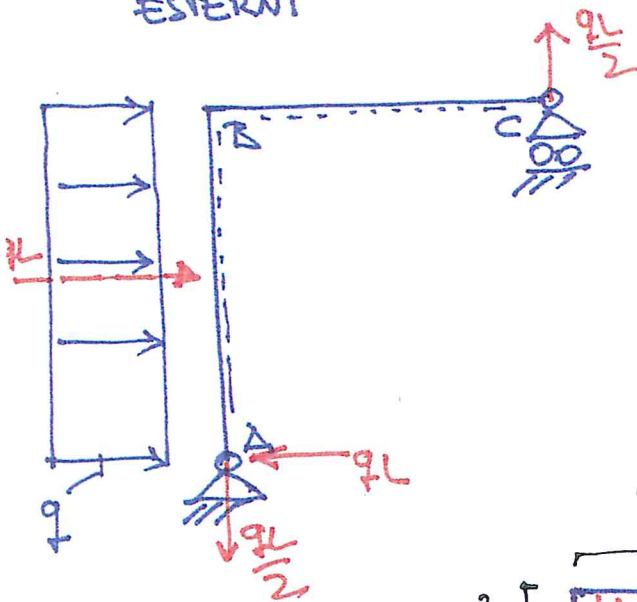


$X_1$

$$\eta_c = -\varepsilon R_{xc} = -\varepsilon X_1$$



## SCHEMA [0] SOLO CARICHI ESTERNI



TRATTO AB  $0 \leq z \leq L$

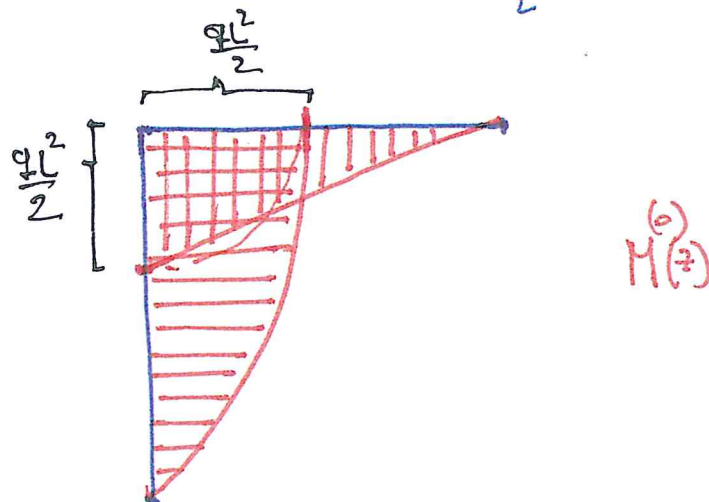
$$\left\{ \begin{array}{l} M_A = 0 \\ M_B = \frac{qL^2}{2} \end{array} \right.$$

$$M^{(0)}(z) = qLz - \frac{qz^2}{2}$$

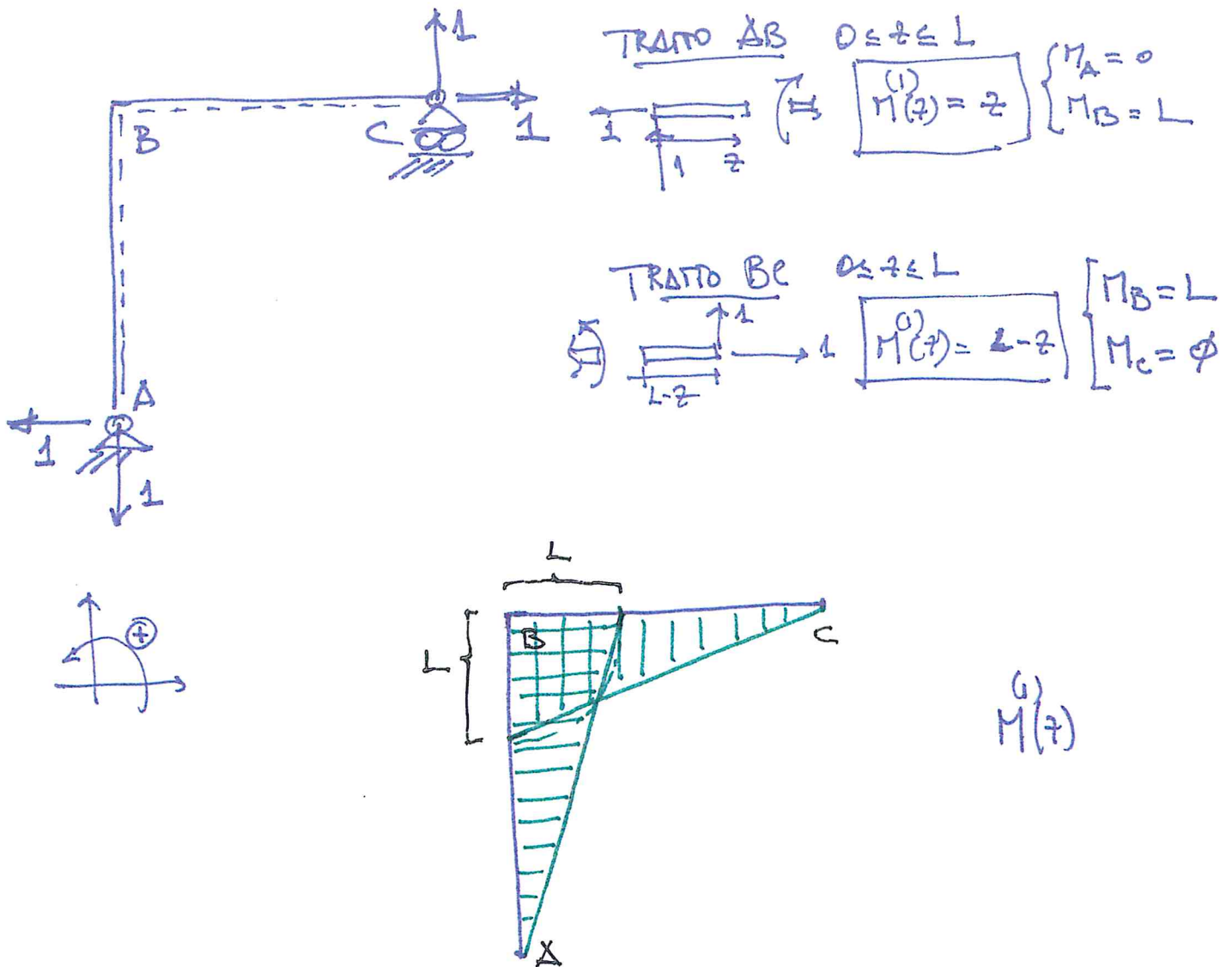
TRATTO BC  $0 \leq z \leq L$

$$\left\{ \begin{array}{l} M_B = \frac{qL^2}{2} \\ M_C = 0 \end{array} \right.$$

$$M^{(0)}(z) = \frac{qL}{2}(L-z)$$



SCHEMA [1]  
solo  $X_1 = 1$



L'unica equazione di Müller-Breslau, corrispondente all'unica incognita iperstatica  $X_1$ , si scrive nella forma  $L_{vz} = L_{vi}$  assumendo come sistema fittizio o lavorante lo schema [1] e come sistema reale la struttura iperstatica data. Si ha:

$$L_{vz} = X_i^{(f)} \eta_i^{(r)} + \sum_j R_j^{(f)} \eta_j^{(r)} = X_1^{(f)} \eta_1^{(r)} + R_{X_A}^{(f)} \eta_A^{(r)} =$$

$$= 1 \cdot (-\varepsilon X_1) + (-1) \eta_A^{(r)} = -\frac{X_1 L^3}{3EI} - \frac{qL^4}{EI}$$

$$L_{vi} = \int_{str} M^{(q)} \frac{M^{(r)}}{EI} dsr + \int_{str} M^{(q)} \frac{\alpha \Delta T}{h} dsr =$$

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(XII)

$$= \int_{str} M^{(1)} \frac{M^{(r)}}{EI} dsr + \int_{str} M^{(1)} \frac{\alpha \Delta T}{h} dsr =$$

$\downarrow$   
 $M^{(b)} + M^{(u)} X_1$

$$= \int_{str} \frac{M^{(1)} M^{(b)}}{EI} dsr + X_1 \int_{str} \frac{[M^{(1)}]^2}{EI} dsr + \int_{str} M^{(1)} \frac{\alpha \Delta T}{h} dsr =$$

$$= \frac{1}{EI} \left\{ \int_{AB} z \left( qLz - \frac{qz^2}{2} \right) dz + \int_{BC} (L-z) \left[ \frac{qL}{2} (L-z) \right] dz \right\} +$$

$$+ \frac{X_1}{EI} \left\{ \int_{AB} z^2 dz + \int_{BC} (L-z)^2 dz \right\} + \int_{BC} (L-z) \frac{\alpha \Delta T}{h} dz =$$

$$= \frac{1}{EI} \left\{ \int_0^L \left[ qLz^2 - \frac{qz^3}{2} \right] dz + \int_0^L \left[ \frac{qL^3}{2} + \frac{qL}{2} z^2 - qL^2 z \right] dz \right\} +$$

$$+ \frac{X_1}{EI} \left\{ \int_0^L z^2 dz + \int_0^L \left[ L^2 + z^2 - 2Lz \right] dz \right\} + \frac{\alpha \Delta T}{h} \int_0^L (L-z) dz =$$

$$= \frac{1}{EI} \left\{ qL \left[ \frac{z^3}{3} \right]_0^L - \frac{q}{2} \left[ \frac{z^4}{4} \right]_0^L + \frac{qL^3}{2} [z]_0^L + \frac{qL}{2} \left[ \frac{z^3}{3} \right]_0^L - qL^2 \left[ \frac{z^2}{2} \right]_0^L \right\} +$$

$$+ \frac{X_1}{EI} \left\{ \left[ \frac{z^3}{3} \right]_0^L + L^2 [z]_0^L + \left[ \frac{z^3}{3} \right]_0^L - 2L \left[ \frac{z^2}{2} \right]_0^L \right\} + \frac{\alpha \Delta T}{h} \left\{ L [z]_0^L - \left[ \frac{z^2}{2} \right]_0^L \right\} =$$

$$= \frac{1}{EI} \left\{ \frac{qL^4}{3} - \frac{qL^4}{8} + \frac{qL^4}{2} + \frac{qL^4}{6} - \frac{qL^4}{2} \right\} +$$

$$+ \frac{X_1}{EI} \left\{ \frac{L^3}{3} + \cancel{\frac{L^3}{3}} + \frac{L^3}{3} - \cancel{\frac{L^3}{3}} \right\} + \frac{\alpha \Delta T}{h} \left\{ L^2 - \frac{L^2}{2} \right\} =$$

$$= \frac{9}{24} \frac{qL^4}{EI} + \frac{2L^3}{3EI} X_1 + \frac{\alpha \Delta T L^2}{h 2}$$

In deflection  $L_{v2} = L_{v1}$  for us see:

$$-\frac{X_1 L^3}{3EI} - \frac{qL^4}{EI} = \frac{q}{24} \frac{qL^4}{EI} + \frac{2L^3}{3EI} X_1 + \frac{q\Delta\Gamma L^2}{h \cdot 2}$$

$$L_0 = \frac{qL^4}{2EI} \text{ facendo conto delle posizioni iniziali!}$$

$$X_1 \left[ \frac{2L^3}{3EI} + \frac{L^3}{3EI} \right] = \frac{qL^4}{EI} \left[ -1 - \frac{q}{24} - \frac{1}{2} \right]$$

$$-\frac{24 - 9 - 12}{24} = \frac{45}{24} \cdot \frac{15}{8}$$



$$X_1 = -\frac{15}{8} qL$$

VERSO OPPOSTO A QUELLO  
INDICATO! OK!



cf. con RV su sist. PRINCIPALE  
ISOSTATICO RISOLTO A PAG. VII!  
NEL QUALE APPUNTO  $R_{xc} = \frac{15}{8} qL$   
VERSO SX!

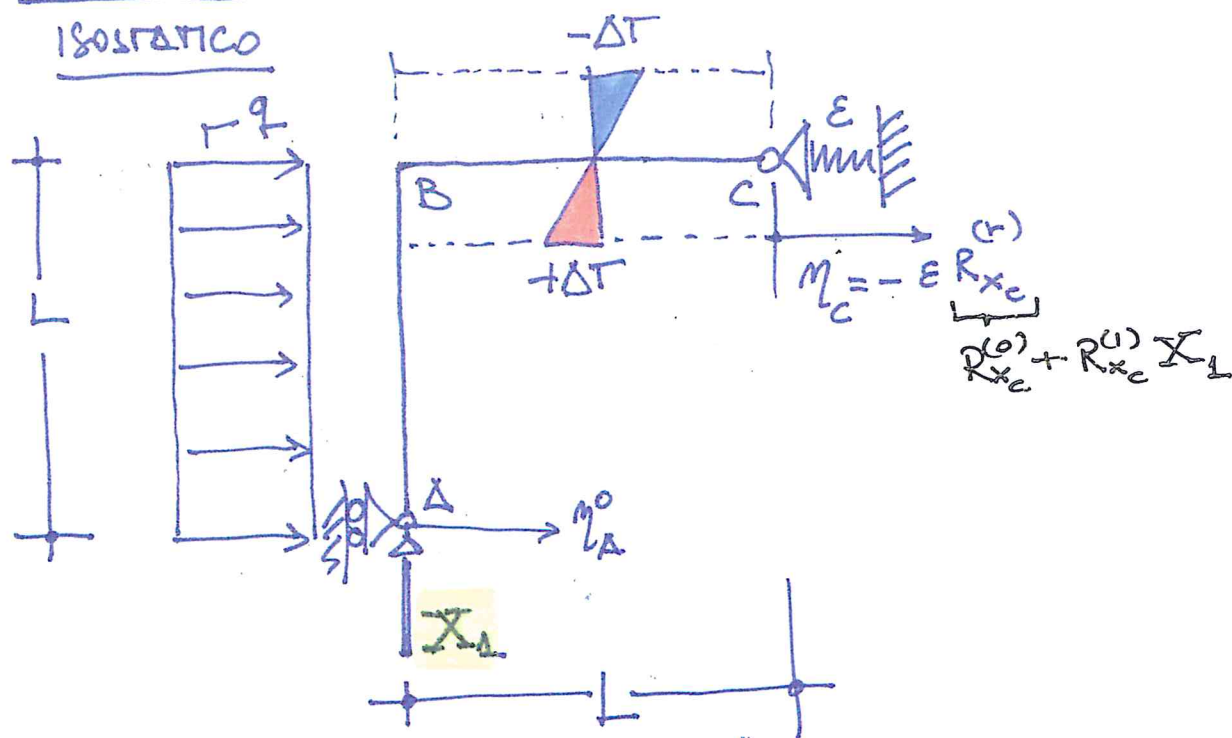
VALGONO OVVIAMENTE I CALCOLI  
DI PAG. VIII!

# SOLUZIONE 3

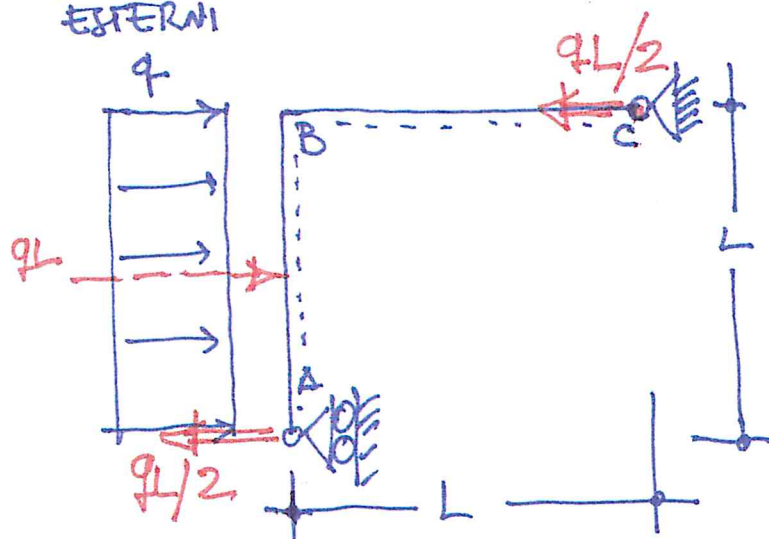
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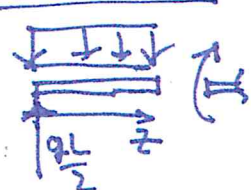
## SISTEMA PRINCIPALE ISOSTATICO



## SCHEMA [0] SOLO CARICHI ESTERNI



### TRATTO AB $0 \leq z \leq L$

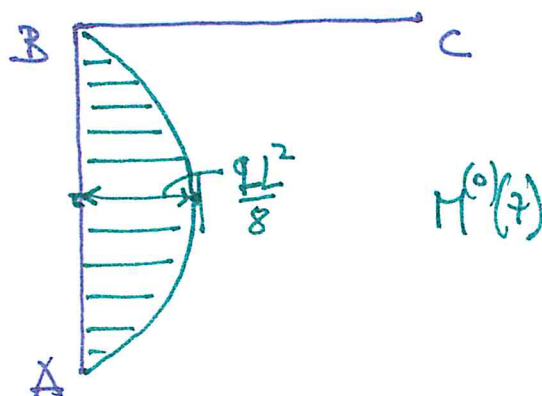
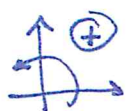


$$M^{(0)} = \frac{qL}{2}z - \frac{qz^2}{2} \quad \left\{ \begin{array}{l} M_A = 0 \\ M_B = 0 \end{array} \right.$$

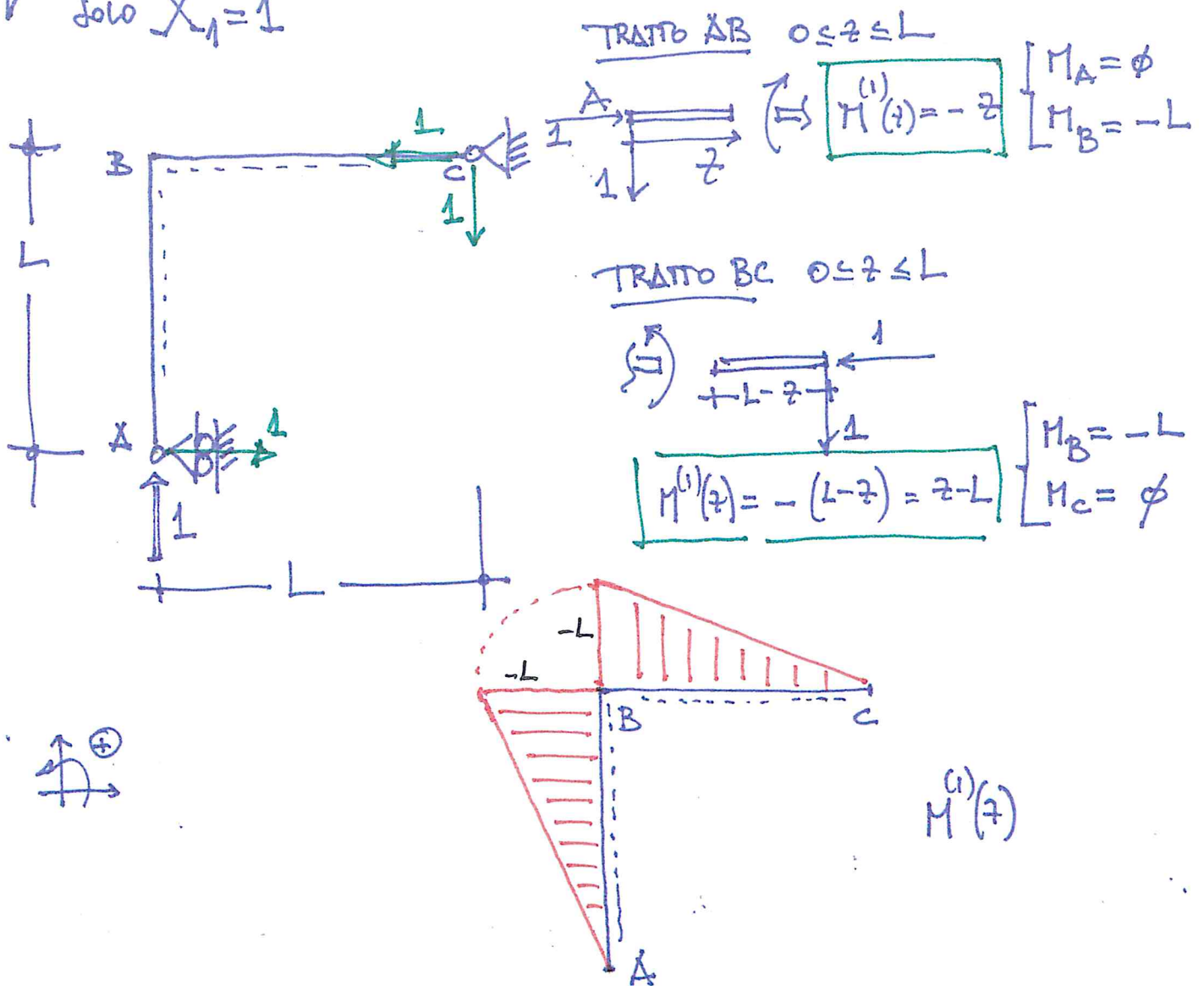
### TRATTO BC

$$M^{(0)}(z) = 0$$

$$M|_{z=L/2} = \frac{qL^2}{4} - \frac{qL^2}{8} = \frac{qL^2}{8}$$



SCHEMA [1]  
solo  $X_1 = 1$



L'unica equazione di Müller-Breslau, corrispondente all'unica incognita iperstatica  $X_1$ , si scrive nelle forme  $Lv_e = Lvi$  considerando come sistema fittizio o lavorante lo schema [1] e come sistema reale le strutture iperstatiche date. Si ha:

$$\begin{aligned}
 Lv_e &= \sum_i X_i^{(f)} \cdot \eta_i^{(r)} + \int_j R_j^{(f)} \eta_j^{(r)} = \underbrace{R_{x_e}^{(0)} + R_{x_c}^{(1)} X_1}_{-ER_{x_e}^{(f)}} = -\frac{qL}{2} - X_1 \\
 &= 1 \cdot \phi + \underbrace{R_{x_A}^{(1)} \cdot \eta_A^0}_{1} + \underbrace{R_{x_c}^{(1)} \cdot \eta_c^{(r)}}_{-1} = \underbrace{1}_{\text{Concorrenza}} - \underbrace{1}_{-1} = -E \left[ \frac{qL}{2} + X_1 \right]
 \end{aligned}$$

Tenendo conto delle posizioni iniziali può scrivere:

XV

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A. PISANO

$$L_{ve} = \underbrace{\frac{qL^4}{EI}}_{\frac{qL^4}{EI}} - \underbrace{\frac{L^3}{3EI}}_{\frac{L^3}{3EI}} \left[ \frac{qL}{2} + X_1 \right] = \frac{qL^4}{EI} - \frac{L^3}{3EI} \left[ \frac{qL}{2} + X_1 \right] =$$

$$= \frac{qL^4}{EI} - \frac{qL^4}{6EI} - \frac{L^3}{3EI} X_1 = \frac{5}{6} \frac{qL^4}{EI} - \frac{L^3}{3EI} X_1$$

$$L_{vi} = \int_{str} \frac{M^{(1)} M^{(0)}}{EI} dstr + \int_{str} \frac{M^{(1)} \alpha \Delta T}{h} dstr =$$

$$= \int_{str} \frac{M^{(1)} M^{(r)}}{EI} dstr + \int_{str} \frac{M^{(1)} \alpha \Delta T}{h} dstr =$$

$\downarrow$   
 $M^{(r)} = M^{(0)} + M^{(1)} X_1$

$$= \int_{str} \frac{M^{(1)} M^{(0)}}{EI} dstr + \left[ \int_{str} \frac{[M^{(1)}]^2}{EI} dstr \right] X_1 + \int_{str} \frac{M^{(1)} \alpha \Delta T}{h} dstr =$$

$$= \frac{1}{EI} \int_{AB} [-z] \left[ \frac{qL}{2} z - \frac{qz^2}{2} \right] dz +$$

$$+ \frac{X_1}{EI} \left[ \int_{AB} z^2 dz + \int_{BC} (z-L)^2 dz \right] + \int_{BC} (z-L) \frac{\alpha \Delta T}{h} dz =$$

$$= \frac{1}{EI} \left\{ \int_0^L \left( -\frac{qL}{2} z^2 + \frac{qz^3}{2} \right) dz \right\}$$

$$+ \frac{X_1}{EI} \left\{ \int_0^L z^2 dz + \int_0^L (z^2 + L^2 - 2Lz) dz \right\} + \frac{\alpha \Delta T}{h} \int_0^L (z-L) dz =$$

$$= \frac{1}{EI} \left\{ -\frac{qL}{2} \left[ \frac{z^3}{3} \right]_0^L + \frac{q}{2} \left[ \frac{z^4}{4} \right]_0^L \right\} + \frac{X_1}{EI} \left\{ \left[ \frac{z^3}{3} \right]_0^L + \left[ \frac{z^3}{3} \right]_0^L + L^2 \left[ z \right]_0^L - 2L \left[ \frac{z^2}{2} \right]_0^L \right\}$$

$$+ \frac{\alpha \Delta T}{h} \left\{ \left[ \frac{z^2}{2} \right]_0^L - L \left[ z \right]_0^L \right\} =$$

$$= \frac{1}{EI} \left\{ \underbrace{-\frac{qL^4}{6} + \frac{qL^4}{8}}_{\frac{-4+3}{24}} \right\} + \frac{X_1}{EI} \left\{ \underbrace{\frac{L^3}{3} + \frac{L^3}{3}}_{\frac{2}{3}L^3} + \cancel{\frac{L^3}{3}} - \cancel{\frac{L^3}{3}} \right\} + \frac{\alpha \Delta T}{h} \left\{ \underbrace{\frac{L^2}{2} - L^2}_{-\frac{L^2}{2}} \right\} =$$

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$$= -\frac{qL^4}{24EI} + \frac{2X_1L^3}{3EI} - \frac{qL^4}{2EI}$$

$\rightarrow = \frac{qL^2}{EI}$  tenuto conto delle posizioni iniziali!

In definitiva  $L_{re} = L_{ri}$  fornisce:

$$\frac{5}{6} \frac{qL^4}{EI} - \frac{L^3}{3EI} X_1 = -\frac{qL^4}{24EI} + \frac{2X_1L^3}{3EI} - \frac{qL^4}{2EI}$$

$$X_1 \left\{ \underbrace{\frac{2}{3} \frac{L^3}{EI} + \frac{L^3}{EI}}_{\frac{L^3}{EI}} \right\} = \frac{qL^4}{EI} \left\{ \underbrace{\frac{5}{6} + \frac{1}{24} + \frac{1}{2}}_{\frac{20+1+12}{24} = \frac{33}{24} = \frac{11}{8}} \right\}$$



$$X_1 = \frac{11}{8} qL$$

POSITIVA  $\rightarrow$  VERSO ROTAZIONE CORRETTA!



- cfr. RV di pag. VII!
- valgono le considerazioni di pag. VIII!